Chapter review

1 a Let the density of the solid be ρ.

Shape	Mass	Mass ratios	Distance of centre of mass from C
Cylinder	$\pi\rho r^2 \times kr$	k	$-\frac{kr}{2}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$\frac{2}{3}$	$\frac{3}{8}r$
Composite body	$\pi\rho r^3\left(k+\frac{2}{3}\right)$	$k+\frac{2}{3}$	0

$$\circlearrowleft \text{M (about } C\text{)}: k\times\left(-\frac{kr}{2}\right)+\frac{2}{3}\times\frac{3}{8}r=0$$

$$\therefore \frac{k^2r}{2}=\frac{r}{4}$$

$$\therefore k^2=\frac{1}{2}\Rightarrow k=\frac{1}{\sqrt{2}}=0.707\ (3\text{ s.f.})$$

b The centre of mass of the body is at C which is always directly above the contact point.

2 a $$\bar{y}=\frac{\rho\int\frac{1}{2}y^2\,dx}{\rho\int y\,dx}=\frac{\frac{1}{2}\int_0^4\frac{x^2}{16}\left(16-8x+x^2\right)dx}{\frac{1}{4}\int_0^4 4x-x^2\,dx}$$

$$=\frac{\frac{1}{2}\int_0^4 x^2-\frac{1}{2}x^3+\frac{1}{16}x^4\,dx}{\frac{1}{4}\left[2x^2-\frac{1}{3}x^3\right]_0^4}$$

$$=2\frac{\left[\frac{1}{3}x^3-\frac{1}{8}x^4+\frac{1}{80}x^5\right]_0^4}{32-\frac{64}{3}}$$

$$=\frac{6}{32}\left[\frac{64}{3}-32+\frac{64}{5}\right]$$

$$=\frac{6}{32}\times\frac{32}{15}$$

$$=\frac{6}{15}=\frac{2}{5}$$

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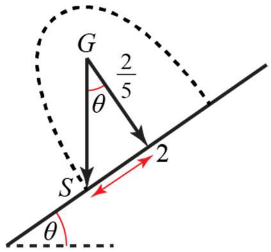
2 b From symmetry the x-coordinate of the centre of mass is 2.

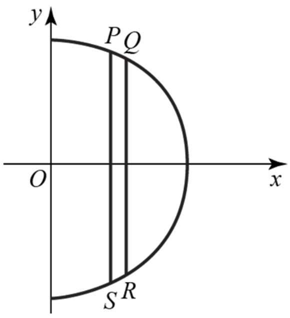
When P is about to topple the centre of mass G directly above the lower edge of the prism S.



$$\therefore \tan\theta = \frac{2}{\frac{2}{5}} = 5$$

$$\therefore \theta = 79° \text{ (nearest degree)}$$

3 a



Take the diameter as the y-axis and the midpoint of the diameter as the origin.

Then $M\bar{x} = \rho\int 2yx\,dx$ where

$$M = \frac{1}{2}\rho\pi(2a^2) \text{ and where } x^2 + y^2 = (2a)^2$$

$$\therefore 2\rho\pi\alpha^2\bar{x} = \rho\int_0^{2a} 2x\sqrt{4a^2 - x^2}\,dx$$

$$= \frac{-2\rho}{3}\left[(4a^2 - x^2)^{\frac{3}{2}}\right]_0^{2a}$$

$$\therefore 2\rho\pi a^2\bar{x} = \frac{2\rho}{3} \times 8a^3$$

$$\therefore \bar{x} = \frac{16}{3}a^3 \div 2\pi a^2$$

$$= \frac{8a}{3\pi}$$

b

Shape	Mass	Mass ratios	Centre of mass (distance from AB)
Large semicircle	$2\pi\rho a^2$	4	$\frac{8a}{3\pi}$
Semicircle diameter AD	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Semicircle diameter OB	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Remainder	$\pi\rho a^2$	2	$\bar{x}$

$$\circlearrowleft MO: 4 \times \frac{8a}{3\pi} - 1 \times \frac{4a}{3\pi} - 1 \times \frac{4a}{3\pi} = 2\bar{x}$$

$$\therefore \frac{24a}{3\pi} = 2\bar{x}$$

$$\therefore \bar{x} = \frac{4a}{\pi}$$

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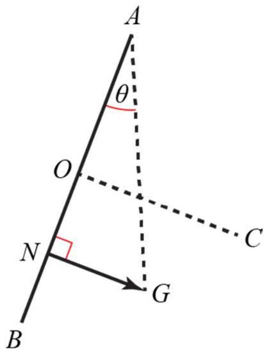
3 c The distance from OC is a

The distance from OB is $\frac{2a}{\pi}$

d



Let N be the foot of the perpendicular from G onto AB.
In the diagram θ is the angle between AB and the vertical.
From ΔANG

$$\tan\theta = \frac{NG}{AN} = \frac{\frac{2a}{\pi}}{2a+a}$$
$$= \frac{2}{3\pi}$$

$\therefore \theta = 12°$ (to the nearest degree)

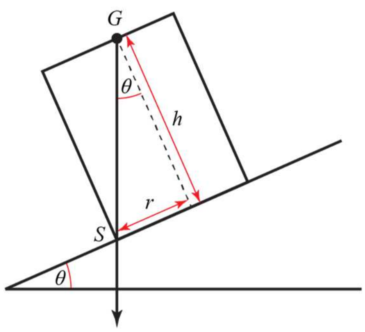
$\therefore$ The angle between AB and the horizontal is $90 - 12 = 78°$ (to the nearest degree)

4 a

Shape	Mass	Mass ratios	Distance of centre of mass from O
Cylinder	$\pi\rho r^2 h$	h	$-\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi\rho(3r)^3$	$18r$	$\frac{3}{8}(3r)$
Mushroom	$\pi\rho r^2(h+18r)$	$h+18r$	0

$$\circlearrowleft M(O): -h\times\frac{h}{2} + 18r\times\frac{3}{8}\times 3r = 0$$
$$\therefore \frac{h^2}{2} = \frac{81r^2}{4}$$
$$\therefore h = r\sqrt{\frac{81}{2}}$$

b



When the mushroom is about to topple GS is vertical.

From the diagram $\tan\theta = \frac{r}{h}$

$$= \sqrt{\frac{2}{81}}$$

$\therefore \theta = 9°$ (nearest degree)

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5 a $V = \pi \int y^2 \, dx$

$= \pi \int_0^a 4ax \, dx$

$= \pi \left[2ax^2 \right]_0^a$

$= 2\pi a^3$

b $\bar{x} = \frac{\pi \int xy^2 dx}{\pi \int y^2 dx}$

$= \frac{\pi \int_0^a 4ax^2 dx}{2\pi a^3}$

$= \pi \frac{\left[\frac{4ax^3}{3} \right]_0^a}{2\pi a^3}$

$= \frac{\frac{4}{3}\pi a x^4}{2\pi a^3}$

$= \frac{2}{3}a$

c

Shape	**Mass**	**Mass ratios**	**Distance of centre of mass from *X***
S_1	$2\pi\rho a^3$	ρ_1	$-\frac{a}{3}$
S_2	$\frac{2}{3}\pi\rho_2(2a)^3$	$\frac{8}{3}\rho_2$	$\frac{3}{8}(2a)$
Combined solid	$2\pi a^3(\rho_1 + \frac{8}{3}\rho_2)$	$\rho_1 + \frac{8}{3}\rho_2$	0

X is the centre of the common plane base.

$\circlearrowleft \mathrm{M}(\mathrm{X})$:

$-\rho_1 \times \frac{a}{3} + \frac{8}{3}\rho_2 \times \frac{6a}{8} = 0$

$\therefore \frac{1}{3}\rho_1 = 2\rho_2$

$\therefore \rho_1 = 6\rho_2$

$\rho_1 : \rho_2 = 6:1$

d Given that $\rho_1 : \rho_2 = 6:1$, then as centre of mass is at centre of hemisphere this will always be above the point of contact with the plane when a point of the curved surface area of the hemisphere is in contact with a horizontal plane.
(Tangent – radius property)

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6 a

Shape	Mass	Mass ratios	Distance of centre of mass from *AB*
Cylinder	$\pi\rho(2r)^2 \times 3r$	$12r$	$\frac{3r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times h$	$\frac{1}{3}h$	$\frac{1}{4}h$
Remainder	$\pi\rho(12r^3 - \frac{1}{3}r^3h)$	$12r - \frac{1}{3}h$	$\bar{x}$

$$\circlearrowleft: \left(12r - \frac{1}{3}h\right)\bar{x} = 12r \times \frac{3r}{2} - \frac{1}{3}h \times \frac{1}{4}h$$

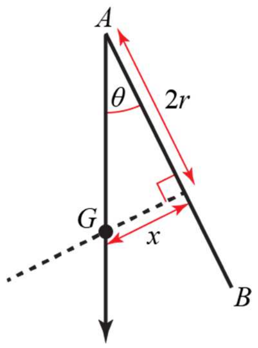
$$\therefore \left(12r - \frac{1}{3}h\right)\bar{x} = 18r^2 - \frac{1}{12}h^2$$

$$\therefore \bar{x} = \frac{18r^2 - \frac{1}{12}h^2}{12r - \frac{1}{3}h}$$

Multiply numerator and denominator by 12

$$\therefore \bar{x} = \frac{216r^2 - h^2}{4(36r - h)}$$

b



From the diagram

$$\tan\theta = \frac{\bar{x}}{2r}$$

As $h = 2r, \bar{x} = \frac{216r^2 - (2r)^2}{4(36r - 2r)} = \frac{212r^2}{136r} = \frac{53}{34}r$

$$\therefore \tan\theta = \frac{53}{68}$$

$\therefore \quad \theta = 38°$ (nearest degree)

7 a First find the centre of mass of the frustum. The centre of mass of the full cone is $\frac{1}{4}h$ from its base, on the symmetry axis. Here h is the height of the full cone, which can be found using similar triangles $\frac{10}{h} = \frac{5}{h-30} \Rightarrow h = 60$ cm. Now taking moments about the base of the cone $\frac{1}{3}\pi 10^2 h \times \frac{1}{4}h - \frac{1}{3}\pi 5^2(h-30)\times\left(\frac{1}{4}h - \frac{1}{4}30 + 30\right)$ $= \frac{1}{3}\pi\left(10^2 h - 5^2(h-30)\right)\bar{x} \Rightarrow \bar{x} = \frac{165}{14}$ cm. Now take moments about the centre of the common plane of the frustum and the solid hemisphere $\frac{1}{3}\pi\rho\left(10^2\times 60 - 5^2 \times 30\right)\bar{x} - 3\rho\frac{2}{3}\pi 10^3 \times \frac{3}{8}\times 10 =$ $\left(\frac{1}{3}\pi\rho\left(10^2 \times 60 - 5^2\times 30\right) + 3\rho\frac{2}{3}\pi 10^3\right)\bar{X} \Rightarrow \bar{X} = 3.5$ cm below their common plane. Thus, the centre of mass of the compound solid is 26.5 cm from its base.

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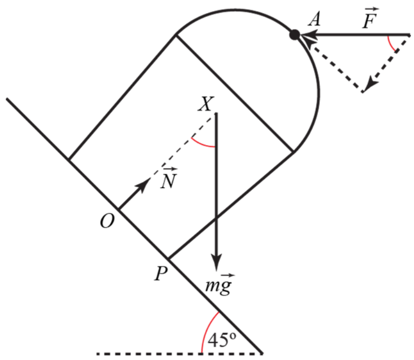
7 b We want to take moments about the lowest base point P, using the fact that distance OP is 5 cm, OA is 40 cm and OX is 26.5 cm.



There are three forces acting on the body, namely $\vec{F}$, $\vec{N}$ and $\overrightarrow{mg}$. At the point of toppling, the reaction force is acting through the point P. It is easiest to decompose the forces into directions parallel and normal to the plane $F\sin 45^\circ \times 40 + F\cos 45^\circ \times 5 + mg\cos 45^\circ \times 5 = mg\sin 45^\circ \times 26.5$

$$\frac{mg}{\sqrt{2}}\times 5+\frac{F}{\sqrt{2}}\times 5+\frac{F}{\sqrt{2}}\times 40=\frac{mg}{\sqrt{2}}\times 26.5 \ \Rightarrow F=\frac{26.5-5}{40+5}mg \approx 0.478mg \ \text{(3 s.f.)}$$

8 a The mass is $M=\rho\int_2^4 \pi y^2\,\mathrm{d}x=\rho\int_2^4 \pi\left(\frac{2}{x+3}\right)^2\mathrm{d}x$

$$=4\pi\rho\left[-\frac{1}{x+3}\right]_2^4=\frac{8}{35}\pi\rho$$

The centre of mass $M\,\bar{x}=\rho\int_2^4 \pi x y^2\,\mathrm{d}x=4\rho\int_2^4 \pi\frac{x}{(x+3)^2}\mathrm{d}x$

$$=4\rho\pi\int_2^4 \frac{x}{(x+3)^2}\mathrm{d}x$$

$$=4\rho\pi\int_2^4\left(\frac{1}{x+3}-\frac{3}{(x+3)^2}\right)\mathrm{d}x$$

$$=4\rho\pi\left(\left[\ln(x+3)\right]_2^4-3\int_2^4\frac{1}{(x+3)^2}\mathrm{d}x\right)$$

$$=4\rho\pi\left[\ln(x+3)+\frac{3}{x+3}\right]_2^4$$

$$=4\rho\pi\left(-\frac{6}{35}+\ln\frac{7}{5}\right)\ \Rightarrow \bar{x}\approx 2.89.$$

Thus, the centre of mass of the solid above the ground is 1.11.

b The radius of the smaller circular end is $y(x=4)=\frac{2}{7}$

The angle at the point of tipping is $\tan\theta=\frac{\frac{2}{7}}{4-\bar{x}}\approx 0.2570 \Rightarrow \theta=14.4^\circ$ (3 s.f.).

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Challenge

a Let the mass per unit volume of the solids be ρ. Let O be the centre of the plane circular faces which coincide.

Shape	Mass	Ratio of masses	Distance of centre of mass from *O*
Cone	$\frac{1}{3}\pi\rho r^2 h$	h	$\frac{h}{4}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$\frac{-3r}{8}$
Toy	$\frac{1}{3}\pi\rho\left(r^2h+2r^3\right)$	$h+2r$	$\bar{x}$

$$\circlearrowleft O\quad (h+2r)\bar{x} = h\times\frac{h}{4} + 2r\left(\frac{-3r}{8}\right)$$

$$= \frac{h^2}{4} - \frac{3r^2}{4}$$

$$\therefore \bar{x} = \frac{\left(h^2-3r^2\right)}{4(h+2r)}$$

b i If $h > r\sqrt{3}$ then $\bar{x} > 0$ so the centre of mass is in the cone – the cone will fall over.

ii If $h < r\sqrt{3}$ then $\bar{x} < 0$ so the centre of mass is in the hemisphere, the toy will return to vertical position.

iii If $h = r\sqrt{3}$, then $\bar{x} = 0$ so the centre of mass is on the join at point O. The toy will remain in equilibrium in its new position.

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